Math 05 – Chapter 5 Test – Spring '11 Name_______________________________

Show all work for credit. Write all responses on separate paper. Do not abuse a calculator.

1. Suppose a parabola has vertex at (0,0) and the focal diameter extends from (1,–4) to (1,4).
   a. Where is the focus?
   b. Where is the directrix?
   c. Write an equation for the parabola.
   d. Construct a careful graph showing the parabola, focus, directrix and focal diameter.

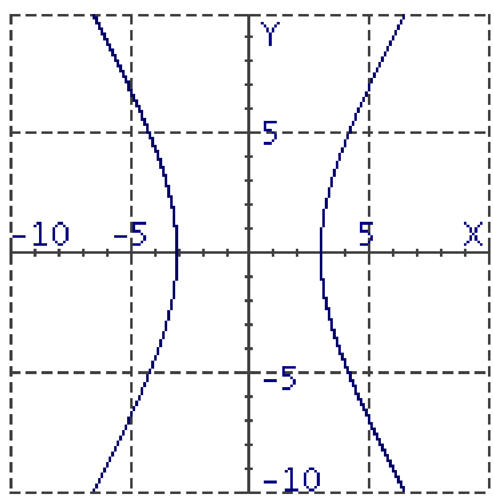
2. A hyperbola has asymptotes $y = \pm 2(x - 3)$ and a vertex at (3,6).
   a. Find an equation for the hyperbola.
   b. Where are the foci?
   c. Construct a careful graph showing the hyperbola together with its asymptotes and foci.

3. Write each of the following conic section equations in standard form. Hint: complete squares for x and y and balance equations.
   a. $y^2 = 3(x - 2y)$
   b. $3x^2 + 2y^2 - 6x - 8y + 5 = 0$
   c. $y^2 - 4x^2 - 2y + 16x = 19$

4. Suppose an ellipse has a vertex at (–8,0) with foci at (0,0) and (4,0).
   a. Plot these three points on the x-axis and indicate an interval of length a and an interval of length c, where a is the distance from the center of the ellipse to a vertex and c is the distance from the center of the ellipse to a focus.
   b. Write an equation for the ellipse in polar coordinate form.
   c. Write an equation for the ellipse in parametric form.

5. Write good approximate equations for the hyperbola whose graph is shown at right in
   a. Standard rectangular form.
   b. Parametric form.

6. According to your equations in #6, what horizontal shift will move the hyperbola so a focus is at the origin? What is the polar form of the equation of the hyperbola after that shift?



For each of this problems, show your own personal expression of the solution using a narrative style documenting each step in your derivation of the solution.

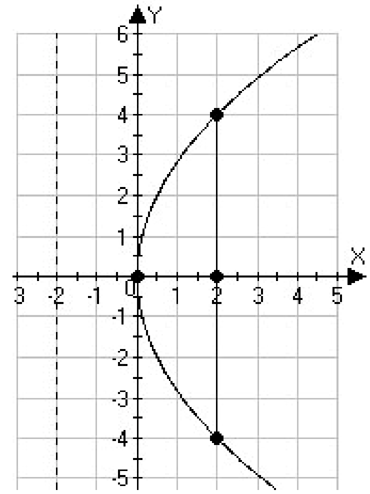
7. Suppose a conic section has a focus at (0,0), eccentricity $e = 2$, and a vertex at (9,0).
   a. Find a standard polar function for the conic.
   b. Find the standard rectangular equation for the conic.
   c. Construct a careful graph of the conic showing key components.
   d. Find parametric equations for the conic.

e. Pick a point P on the conic which is not a vertex and show that the ration of the distance from that point to a focus to the distance from that point to the directrix is $\dfrac{d(P,F)}{d(P,\ell)} = e$

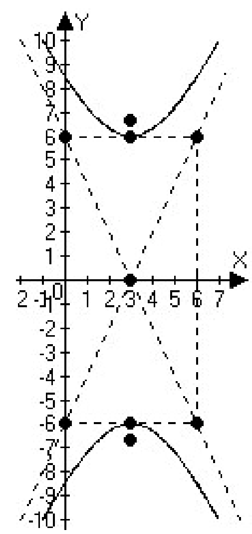
8. Suppose a conic section has a focus at (0,0), eccentricity $e = ½$, and a vertex at (0,-8).
   a. Find a standard polar function for the conic.
   b. Find the standard rectangular equation for the conic.
   c. Construct a careful graph of the conic showing key components.
   d. Find parametric equations for the conic.
   e. Pick a point P on the conic which is not a vertex and show that the ration of the distance from that point to a focus to the distance from that point to the directrix is $\dfrac{d(P,F)}{d(P,\ell)} = e$

Math 05 – Chapter 5 Test Solutions – Spring '11

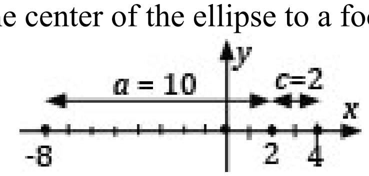
1. Suppose a parabola has vertex at (0,0) and the focal diameter extends from (2,–4) to (2,4).
   a. Where is the focus? SOLN: (2,0)
   b. Where is the directrix? SOLN: $x = -2$
   c. Write an equation for the parabola. $y^2 = 8x$
   d. Construct a careful graph showing the parabola, focus, directrix and focal diameter. SOLN: →



2. A hyperbola has asymptotes $y = \pm 2(x - 3)$ and a vertex at (3,6).
   a. Find an equation for the hyperbola.
      SOLN: The center of the hyperbola is the point of intersection of the asymptotes: (3,0). That means that $a = 6$. Since the slope of the asymptotes is $a/b = 2$, $b = 3$. Thus the equation for the hyperbola is $\frac{y^2}{36} - \frac{(x-3)^2}{9} = 1$
   b. Where are the foci? SOLN: $c^2 = a^2 + b^2 = 36 + 9 = 45$ so $c = \sqrt{45} = 3\sqrt{5} \approx 6.7$
   c. Construct a careful graph showing the hyperbola together with its asymptotes and foci.
      SOLN: →



3. Write each of the following conic section equations in standard form. Hint: complete squares for x and y and balance equations.
   a. $y^2 = 3(x - 2y)$
      SOLN: $y^2 = 3(x - 2y) \leftrightarrow y^2 = 3x - 6y \leftrightarrow y^2 + 6y = 3x$
      $\leftrightarrow y^2 + 6y + 9 = 3x + 9 \leftrightarrow (y + 3)^2 = 3(x+3)$
   b. $3x^2 + 2y^2 - 6x - 8y + 5 = 0 \leftrightarrow 3(x^2 - 2x) + 2(y^2 - 4y) = -5$
      $\leftrightarrow 3(x-1)^2 + 2(y-2)^2 = 6$ So $\boxed{\frac{(x-1)^2}{2} + \frac{(y-2)^2}{3} = 1}$
   c. $y^2 - 4x^2 - 2y + 16x = 19 \leftrightarrow y^2 - 2y - 4(x^2 - 4x) = 19 \leftrightarrow (y - 1)^2 - 4(x - 2)^2 = 4$
      So $\boxed{\frac{(y-1)^2}{4} - (x-2)^2 = 1}$
4. Suppose an ellipse has a vertex at (–8,0) with foci at (0,0) and (4,0).
   a. Plot these three points on the x-axis and indicate an interval of length a and an interval of length c, where a is the distance from the center of the ellipse to a vertex and c is the distance from the center of the ellipse to a focus.

      SOLN: 

b. Write an equation for the ellipse in polar coordinate form.

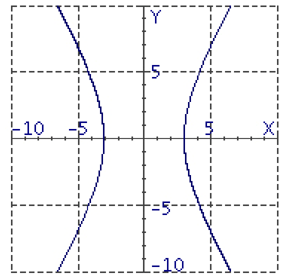
SOLN: The eccentricity is $c/a = 1/5 = 0.2$ so the standard form is $r = \frac{0.2d}{1 - 0.2\cos\theta}$.

Since $(r,\theta) = (8,\pi)$ we have $8 = \frac{0.2d}{1 - 0.2\cos\pi} = \frac{d}{6} \Rightarrow d = 48$ so $r = \frac{9.6}{1 - 0.2\cos\theta} = \frac{48}{5 - \cos\theta}$

c. Write an equation for the ellipse in parametric form.

SOLN: $a^2 = b^2 + c2$ so $b^2 = 100 - 4 = 96$ and so $\boxed{\begin{array}{l} x = 2 + 10\cos t \\ y = 4\sqrt{6}\sin t \end{array}}$

5. The hyperbola whose graph is shown at right at vertices at (±3,0) and passes through (4,5).



a. Standard rectangular form. SOLN: $\frac{x^2}{9} - \frac{y^2}{b^2} = 1$ must hold at (4,5) so

$\frac{16}{9} - \frac{25}{b^2} = 1 \Leftrightarrow \frac{25}{b^2} = \frac{7}{9} \Leftrightarrow b^2 = \frac{225}{7}$ so $\boxed{\frac{x^2}{9} - \frac{7y^2}{225} = 1}$

b. Parametric form. SOLN: $x = 3\sec(t)$ and $y = 15\sqrt{7}\tan(t) / 7$

6. According to your equations in #6, what horizontal shift will move the hyperbola so a focus is at the origin? What is the polar form of the equation of the hyperbola after that shift?

SOLN: $c^2 = a^2 + b^2 = 9 + 225/7 = 288/7$ so $c = \frac{12\sqrt{14}}{7}$ and shifting that far either left or right will move the focus to the origin.

Suppose we shift to the left. The eccentricity is $c/a = 4\sqrt{14}/7$ so the standard form is

$r = \frac{\frac{4\sqrt{14}}{7}d}{1 - \frac{4\sqrt{14}}{7}\cos\theta} = \frac{4\sqrt{14}d}{7 - 4\sqrt{14}\cos\theta}$. Now the vertex that was at (3,0) is now at $x = 3 - \frac{12\sqrt{14}}{7}$ so

the polar coordinates $(r,\theta) = \left(\frac{12\sqrt{14}}{7} - 3, \pi\right)$ are a vertex and solving for the numerator of the

RHS: $\frac{12\sqrt{14}}{7} - 3 = \frac{4\sqrt{14}d/7}{1 - \frac{4\sqrt{14}}{7}\cos\pi} \Leftrightarrow \frac{4\sqrt{14}d}{7} = 3\left(\frac{4\sqrt{14}}{7} - 1\right)\left(1 + \frac{4\sqrt{14}}{7}\right) = 3\left(\frac{16\cdot 14}{49} - 1\right) = \frac{75}{7}$

so the polar form of the hyperbola is $r = \frac{75}{7 - 4\sqrt{14}\cos\theta}$.

For each of these problems, show your own personal expression of the solution using a narrative style documenting each step in your derivation of the solution.

7. Suppose a conic section has a focus at (0,0), eccentricity $e = ½$, and a vertex at (0,-8).
   a. Find a standard polar function for the conic.

      SOLN: Plugging $e = ½$ into the standard polar form we get $r = \dfrac{d/2}{1-\frac{1}{2}\sin\theta}$

      so at $(r,\theta) = (8,3\pi/2)$, $8 = \dfrac{d/2}{1-\frac{1}{2}\sin\frac{3\pi}{2}} = \dfrac{d}{3} \Rightarrow d = 24$ so $\boxed{r = \dfrac{24/2}{1-\frac{1}{2}\sin\theta} = \dfrac{24}{2-\sin\theta}}$

   b. Find the standard rectangular equation for the conic.

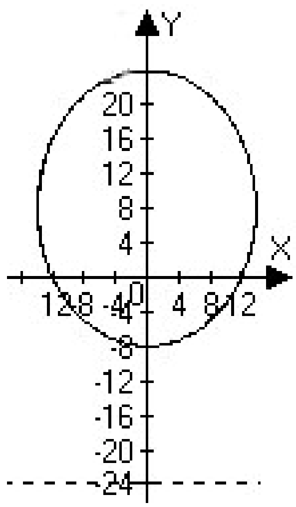
      SOLN: Tabulate solutions:

r	12	24	12	8
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

      to see the center is at (0,8) and that $a = 16$ and $c = 8$ whence $b^2 = a^2 - c^2 = 256 - 64 = 192$.

      The rectangular equation is then $\dfrac{x^2}{192} + \dfrac{(y-8)^2}{256} = 1$

   c. Construct a careful graph of the conic showing key components.



   d. Find parametric equations for the conic.

      SOLN: Plugging into the standard polar form, $x = 8\sqrt{3}\cos(t)$, $y = 8 + 16\sin(t)$

   e. Pick a point P on the conic which is not a vertex and show that the ration of the distance from that point to a focus to the distance from that point to the directrix is $\dfrac{d(P,F)}{d(P,\ell)} = e$

      SOLN Look at the x-intercept at (12,0). Its distance from the focus (12) is half its distance from the directrix (24).

8. Suppose a conic section has a focus at (0,0), eccentricity $e = 2$, and a vertex at (9,0).

a. Find a standard polar function for the conic.

SOLN: Plugging $e = 2$ into the standard form for the conic, we get

$$r = \frac{2d}{1+2\cos\theta} \Rightarrow 9 = \frac{2d}{3} \Leftrightarrow d = \frac{27}{2} \text{ so } \boxed{r = \frac{27}{1+2\cos\theta}}$$

b. Find the standard rectangular equation for the conic.

SOLN: Starting with the polar form, multiply both sides by $1 + 2\cos\theta$ and substitute $x = r\cos\theta$ and $r^2 = x^2 + y^2$ to get $r = \frac{27}{1+2\cos\theta} \Leftrightarrow r + 2x = 27 \Rightarrow x^2 + y^2 = (27-2x)^2 = 729 - 108x + 4x^2$

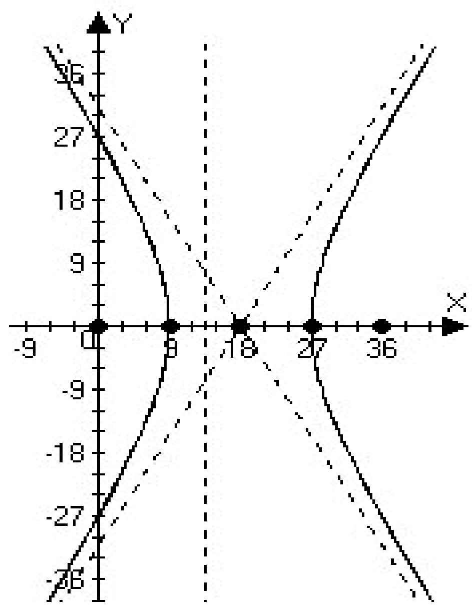
Now complete the square for x to get the rectangular form is $\boxed{\frac{(x-18)^2}{81} - \frac{y^2}{243} = 1}$

Alternatively, tabulate solutions:

r	9	27	−27	27
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

to see the center is at (18,0) and $a = 9$.

c. Construct a careful graph of the conic showing key components.



d. Find parametric equations for the conic. SOLN: Plugging into the standard form $x = h + a\sec(t)$, $y = k + b\tan(t)$, we get $\boxed{\begin{array}{l} x = 18 + 9\sec(t) \\ y = 9\sqrt{3}\tan(t) \end{array}}$.

e. Pick a point P on the conic which is not a vertex and show that the ratio of

(distance to focus) to (distance to directrix)

is $\frac{d(P,F)}{d(P,\ell)} = e$

SOLN: Consider the y-intercept at (0,27). In this case,

$$\frac{d(P,F)}{d(P,\ell)} = \frac{27}{13.5} = 2$$