The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

Recounting the Rationals

Neil Calkin & Herbert S. Wilf

To cite this article: Neil Calkin & Herbert S. Wilf (2000) Recounting the Rationals, The American Mathematical Monthly, 107:4, 360-363, DOI: 10.1080/00029890.2000.12005205

To link to this article: https://doi.org/10.1080/00029890.2000.12005205

Published online: 23 Apr 2018.

Proof: Since $P_n(w) = (2b)_n F(-n, b; 2b; 1 - w)$ and $(2b)_n \neq 0$, we require the values of w for which $P_n(w) = 0$. By Lemma 1, $P_n(w) = 0$ if and only if

$$\frac{d^n}{d\lambda^n}((1 - w\lambda)(1 - \lambda))^{-b} = 0 \quad \text{when } \lambda = 0.$$

By Lemma 3, *the values of* λ *for which the* n-*th derivative is zero all lie on the perpendicular bisector of the line segment joining the points* $\lambda = w^{-1}$ *and* $\lambda = 1$. (*Note that* $w \neq 1$, *since* $P_n(1) = (2b)_n \neq 0$.) *Thus if* $P_n(w) = 0$, *then the point* $\lambda = 0$ *lies on this perpendicular bisector, that is,* w^{-1} *and* 1 *are equidistant from* 0, *so* $|w| = 1$.

A relationship between the zeros of $r_n(x)$ and those of $P_n(w)$ can be found using similarity. For each zero α_j of $r_n(x)$, the triangle with vertices α_j, i, $-i$ in the x-plane is similar to the triangle with vertices 0, w_j^{-1}, 1 respectively, in the λ-plane. It follows easily that $w_j = (\alpha_j + i)/(\alpha_j - i)$, a familiar Möbius transformation from the real line to the unit circle.

ACKNOWLEDGMENTS. I thank Kathy Driver and Peter Duren for so freely discussing their research with me, and Mike Mays and West Virginia University for the visiting appointment that gave me the time to prepare this paper.

REFERENCES

1. P. B. Borwein and W. Chen, Incomplete rational approximation in the complex plane, *Constr. Approx.*, **11** (1995) 85–106.
2. K. A. Driver and P. L. Duren, Asymptotic zero distribution of hypergeometric polynomials, *Numer. Algorithms*, to appear.
3. K. A. Driver and P. L. Duren, Zeros of the hypergeometric polynomials $F(-n, b; 2b; z)$, *Indag. Math. (N.S)*, to appear.

University of the Witwatersrand, Private Bag 3, Wits, South Africa, 2050
036jnr@cosmos.wits.ac.za

Recounting the Rationals

Neil Calkin and Herbert S. Wilf

It is well known (indeed, as Paul Erdős might have said, every child knows) that the rationals are countable. However, the standard presentations of this fact do not give an explicit enumeration; rather they show how to *construct* an enumeration. In this note we explicitly describe a sequence $b(n)$ with the property that every positive rational appears exactly once as $b(n)/b(n+1)$. Moreover, $b(n)$ is the solution of a quite natural counting problem.

Our list of the positive rational numbers begins like this:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}, \frac{8}{5}, \frac{5}{7}, \frac{7}{2}, \frac{2}{7}, \frac{7}{5}, \ldots$$

Some of the interesting features of this list are

1. The denominator of each fraction is the numerator of the next one. That means that the nth rational number in the list looks like $b(n)/b(n+1)$ $(n = 0, 1, 2, \ldots)$, where b is a certain function of the nonnegative integers

whose values are

$$\{b(n)\}_{n\geq 0} = \{1,1,2,1,3,2,3,1,4,3,5,2,5,3,4,1,5,4,7,\dots\}.$$

2. The function values $b(n)$ actually count something nice. In fact, $b(n)$ is the number of ways of writing the integer n as a sum of powers of 2, each power being used at most twice (i.e., once more than the legal limit for binary expansions). For instance, we can write $5 = 4 + 1 = 2 + 2 + 1$, so there are two such ways to write 5, and therefore $b(5) = 2$. Let's say that $b(n)$ is the number of *hyperbinary* representations of the integer n.
3. Consecutive values of this function b are always relatively prime, so that each rational occurs in reduced form when it occurs.
4. Every positive rational occurs once and only once in this list.

1 The tree of fractions. For the moment, let's forget about enumeration, and just imagine that fractions grow on the tree that is completely described, inductively, by the following two rules:

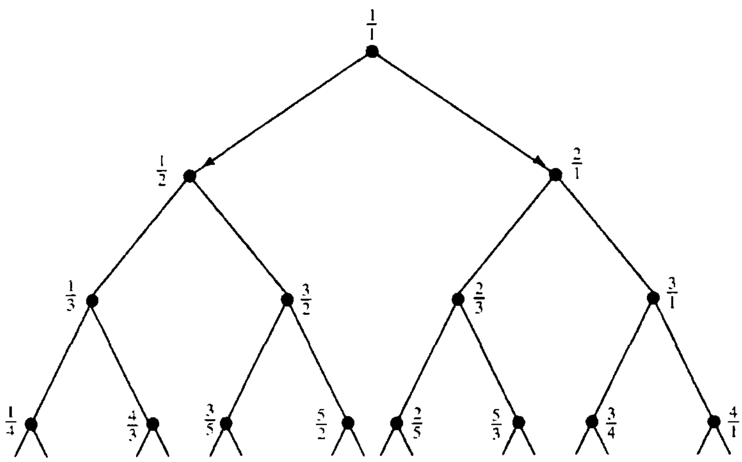

Figure 1. The tree of fractions

- $\frac{1}{1}$ is at the top of the tree, and
- Each vertex $\frac{i}{j}$ has two children: its left child is $\frac{i}{i+j}$ and its right child is $\frac{i+j}{j}$.

We show the following properties of this tree.

1. *The numerator and denominator at each vertex are relatively prime.* This is certainly true at the top vertex. Otherwise, suppose r/s is a vertex on the highest possible level of the tree for which this is false. If r/s is a left child, then its parent is $r/(s-r)$, which would clearly also not be a reduced fraction, and would be on a higher level, a contradiction. If r/s is a right child, then its parent is $(r-s)/s$, which leads to the same contradiction. ■
2. *Every reduced positive rational number occurs at some vertex.* The rational number 1 certainly occurs. Otherwise, let r/s be, among all fractions that do not occur, one of smallest denominator, and among those the one of smallest numerator. If $r > s$ then $(r-s)/s$ doesn't occur either, else one of its children would be r/s, and its numerator is smaller, the denominator being the same, a contradiction. If $r < s$, then $r/(s-r)$ doesn't occur

either, else one of its children would be r/s, and it has a smaller denominator, a contradiction. ■

3. *No reduced positive rational number occurs at more than one vertex.* First, the rational number 1 occurs only at the top vertex of the tree, for if not, it would be a child of some vertex r/s. But the children of r/s are $r/(r+s)$ and $(r+s)/s$, neither of which can be 1. Otherwise, among all reduced rationals that occur more than once, let r/s have the smallest denominator, and among these, the smallest numerator. If $r < s$ then r/s is a left child of two distinct vertices, at both of which $r/(s-r)$ lives, contradicting the minimality of the denominator. The case $r > s$ is similar. ■

It follows that a list of all positive rational numbers, each appearing once and only once, can be made by writing down 1/1, then the fractions on the level just below the top of the tree, reading from left to right, then the fractions on the next level down, reading from left to right, etc.

We claim that if that be done, then the denominator of each fraction is the numerator of its successor. This is clear if the fraction is a left child and its successor is the right child of the same parent. If the fraction is a right child then its denominator is the same as the denominator of its parent and the numerator of its successor is the same as the numerator of the parent of its successor, hence the result follows by downward induction on the levels of the tree. Finally, the rightmost vertex of each row has denominator 1, as does the leftmost vertex of the next row, proving the claim.

Thus, after we make a single sequence of the rationals by reading the successive rows of the tree as described above, the list will be in the form $\{f(n)/f(n+1)\}_{n \geq 0}$, for some f.

Now, as the fractions sit in the tree, the two children of $f(n)/f(n+1)$ are $f(2n+1)/f(2n+2)$ and $f(2n+2)/f(2n+3)$. Hence from the rule of construction of the children of a parent, it must be that

$$f(2n+1) = f(n) \quad \text{and} \quad f(2n+2) = f(n) + f(n+1) \quad (n = 0, 1, 2, \dots).$$

These recurrences, together with $f(0) = 1$, evidently determine our function f on all nonnegative integers.

We claim that $f(n) = b(n)$, the number of hyperbinary representations of n, for all $n \geq 0$.

This is true for $n = 0$, and suppose it is true for all integers $0, 1, \dots, 2n$. Now $b(2n+1) = b(n)$, because if we are given a hyperbinary expansion of $2n+1$, the "1" must appear, hence by subtracting 1 from both sides and dividing by 2, we'll get a hyperbinary representation of n. Conversely, given such an expansion of n, double each part and add a 1 to obtain a representation of $2n+1$.

Furthermore, $b(2n+2) = b(n) + b(n+1)$, for a hyperbinary expansion of $2n+2$ might have either two 1's or no 1's in it. If it has two 1's, then by deleting them and dividing by 2 we obtain an expansion of n. If it has no 1's, then we just divide by 2 to get an expansion of $n+1$. These maps are reversible, proving the claim.

It follows that $b(n)$ and $f(n)$ satisfy the same recurrence formulas and take the same initial values, hence they agree for all nonnegative integers. We state the final result as follows.

Theorem 1. *The nth rational number, in reduced form, can be taken to be* $b(n)/b(n+1)$, *where* $b(n)$ *is the number of hyperbinary representations of the integer* n, *for*

$n = 0, 1, 2, \ldots$. *That is, $b(n)$ and $b(n+1)$ are relatively prime, and each positive reduced rational number occurs once and only once in the list $b(0)/b(1), b(1)/b(2), \ldots$.*

Remarks. There is a large literature on the closely related subject of Stern-Brocot trees [**1**], [**6**]. In particular, an excellent introduction is in [**2**], and the relationship between these trees and hyperbinary partitions is explored in [**4**]. Our sequence $\{b(n)\}$ is sequence #A002487 in [**5**]. We thank Neil Sloane for pointing out that still other ways of counting the rationals are in his sequences #A038568 and #A020651. Our interest in $\{b(n)\}$ was piqued by a problem in *Quantum*, in September 1997, that asked for $b(90316)$, and which was posted by Stan Wagon as his "Problem of the Week".

In Stern's original paper [**6**] of 1858 there is a structure that is essentially our tree of fractions, though in a different garb, and he proved that every rational number occurs once and only once, in reduced form. However Stern did not deal with the partition function $b(n)$. Reznick [**4**] studied restricted binary partition functions and observed their relationship to Stern's sequence. Nonetheless it seemed to us worthwhile to draw these two aspects together and explicitly note that the ratios of successive values of the partition function $b(n)$ run through all of the rationals.

A question: What other functions $f(n)$ are there that have natural and intuitive definitions and also have the property that $\{f(n)/f(n+1)\}$ takes every rational value exactly once?

REFERENCES

1. Achille Brocot, Calcul des rouages par approximation, nouvelle méthode, *Revue Chrono-métrique* **6** (1860) 186–194.
2. Ronald L. Graham, Donald E. Knuth and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading, 1989.
3. D. H. Lehmer, On Stern's diatomic series, this MONTHLY **36** (1929) 59–67.
4. Bruce Reznick, Some binary partition functions, in *Analytic Number Theory*, Proceedings of a conference in honor of Paul T. Bateman, Birkhäuser, Boston (1990) 451–477.
5. N. J. A. Sloane (1999), The On-Line Encyclopedia of Integer Sequences, published electronically at <http://www.research.att.com/~njas/sequences/>.
6. M. A. Stern, Über eine zahlentheoretische Funktion, *J. Reine Angew. Math.* **55** (1858) 193–220.

Clemson University, Clemson, SC 29634
calkin@math.clemson.edu

University of Pennsylvania, Philadelphia, PA 19104-6395
wilf@math.upenn.edu