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Enumerating the Rationals from Left to Right

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Enumerating the Rationals from Left to Right

S. P. Glasby

Abstract. The rationals can be enumerated using Stern-Brocot sequences $\mathcal{SB}_n$, Calkin-Wilf sequences $\mathcal{CW}_n$, or Farey sequences $\mathcal{F}_n$. We show that the rationals in $\mathcal{SB}_n, \mathcal{CW}_n$, and $\mathcal{F}_n$ can be computed using very similar second-order linear recurrence relations. (This, incidentally, obviates the need to compute previous sequences $\mathcal{SB}_i, \mathcal{CW}_i, \mathcal{F}_i$ with $i < n$.)

1. INTRODUCTION. There are three well-known sequences used to enumerate the rationals: the Stern-Brocot sequences $\mathcal{SB}_n$, the Calkin-Wilf sequences $\mathcal{CW}_n$, and the Farey sequences $\mathcal{F}_n$. The purpose of this note is to show that all three sequences can be constructed (left-to-right) using almost identical recurrence relations. The Stern-Brocot (S-B) and Calkin-Wilf (C-W) sequences give rise to complete binary trees related to Figure 1. These trees have many beautiful algebraic, combinatorial, computational, and geometric properties [**2, 5, 4**]. Well-written introductions to the S-B tree and Farey sequences can be found in [**3**], and to the C-W tree in [**2**]. We shall focus on *sequences* rather than *trees*.

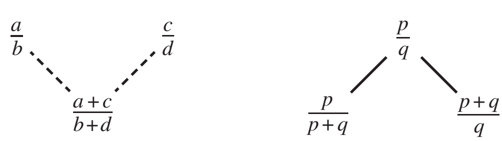

Figure 1. S-B rules (left), and C-W rules (right).

Two fractions $\frac{a}{b} < \frac{c}{d}$ are called *adjacent* if $bc - ad = 1$. Adjacent fractions are necessarily reduced, i.e., $\gcd(a, b) = \gcd(c, d) = 1$. The *mediant* of $\frac{a}{b} < \frac{c}{d}$ is $\frac{a+c}{b+d}$. A short calculation shows that if $\frac{a}{b} < \frac{c}{d}$ are adjacent, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ are pairwise adjacent (and hence reduced). The sequences $\mathcal{SB}_n$ are defined recursively: $\mathcal{SB}_0 = \left[\frac{0}{1}, \frac{1}{0}\right]$ represents 0 and ∞ as reduced fractions, and $\mathcal{SB}_n$ is computed from $\mathcal{SB}_{n-1}$ by inserting mediants between consecutive fractions. Thus

$$\mathcal{SB}_1 = \left[\frac{0}{1}, \frac{\mathbf{1}}{\mathbf{1}}, \frac{1}{0}\right],$$

$$\mathcal{SB}_2 = \left[\frac{0}{1}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{1}{1}, \frac{\mathbf{2}}{\mathbf{1}}, \frac{1}{0}\right],$$

$$\mathcal{SB}_3 = \left[\frac{0}{1}, \frac{\mathbf{1}}{\mathbf{3}}, \frac{1}{2}, \frac{\mathbf{2}}{\mathbf{3}}, \frac{1}{1}, \frac{\mathbf{3}}{\mathbf{2}}, \frac{2}{1}, \frac{\mathbf{3}}{\mathbf{1}}, \frac{1}{0}\right], \ \ldots.$$

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A simple induction shows that $|\mathcal{SB}_n| = 2^n + 1$. Thus 2^{n-1} mediants are inserted into $\mathcal{SB}_{n-1}$ to form $\mathcal{SB}_n$. The C-W sequences are defined using the right rule in Figure 1:

$$\begin{aligned}
\mathcal{CW}_1 &:= \left[\frac{1}{1}\right], \\
\mathcal{CW}_2 &= \left[\frac{1}{2}, \frac{2}{1}\right], \\
\mathcal{CW}_3 &= \left[\frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}\right], \\
\mathcal{CW}_4 &= \left[\frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}\right], \ldots .
\end{aligned}$$

A simple induction shows that $|\mathcal{CW}_n| = 2^{n-1}$. Another simple induction (see [**2**, p. 360]) shows that the fractions in $\mathcal{CW}_n$ have the form

$$\mathcal{CW}_n = \left[\frac{b_{-1}}{b_0}, \frac{b_0}{b_1}, \ldots, \frac{b_{N-2}}{b_{N-1}}\right] \qquad \text{where } N = 2^{n-1},$$

and the denominator of a given fraction is the numerator of the succeeding fraction. Indeed, this property obtains even when the sequences $\mathcal{CW}_1, \mathcal{CW}_2, \mathcal{CW}_3, \ldots$ are concatenated to form

$$\mathcal{CW}_\infty := \left[\frac{1}{\mathbf{1}}, \overbrace{\frac{\mathbf{1}}{2}, \frac{2}{\mathbf{1}}}, \overbrace{\frac{\mathbf{1}}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{\mathbf{1}}}, \overbrace{\frac{\mathbf{1}}{4}, \ldots, \frac{4}{\mathbf{1}}}, \ldots\right] = \left[\frac{c_0}{c_1}, \frac{c_1}{c_2}, \frac{c_2}{c_3}, \ldots\right].$$

The Farey sequence of order n contains all the reduced fractions $\frac{p}{q}$ with $0 \le p \le q \le n$, in their natural order. Thus

$$\begin{aligned}
\mathcal{F}_1 &= \left[\frac{0}{1}, \frac{1}{1}\right], \\
\mathcal{F}_2 &= \left[\frac{0}{1}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{1}{1}\right], \\
\mathcal{F}_3 &= \left[\frac{0}{1}, \frac{\mathbf{1}}{\mathbf{3}}, \frac{1}{2}, \frac{\mathbf{2}}{\mathbf{3}}, \frac{1}{1}\right], \\
\mathcal{F}_4 &= \left[\frac{0}{1}, \frac{\mathbf{1}}{\mathbf{4}}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{\mathbf{3}}{\mathbf{4}}, \frac{1}{1}\right], \ldots .
\end{aligned}$$

A standard way to compute $\mathcal{F}_n$ from $\mathcal{F}_{n-1}$ is to insert mediants between consecutive fractions of $\mathcal{F}_{n-1}$ only when this gives a denominator of size n (see [**3**, p. 118]). Thus $\mathcal{F}_n$ is a subsequence of $\mathcal{SB}_n$. The number of reduced fractions $\frac{a}{n}$ with $1 \le a < n$ is denoted $\varphi(n)$. Counting the nonzero reduced fractions with denominator j, together with zero, gives $|\mathcal{F}_n| = 1 + \sum_{j=1}^{n} \varphi(j)$. The mediant rule above implies that consecutive fractions in $\mathcal{SB}_n$ and $\mathcal{F}_n$ are adjacent (see also [**3**, p. 119]).

2. RECURSIVE RESULTS. It is shown in [**3**] and [**2**] that $\mathcal{SB}_\infty = \bigcup_{n=0}^{\infty} \mathcal{SB}_n$ and $\mathcal{CW}_\infty$ contain *every* (reduced) positive rational *precisely once*. Although $\mathcal{SB}_n, \mathcal{CW}_n$, and $\mathcal{F}_n$ are defined "top-down" they can be computed from "left to right" via almost identical recurrence relations. The 2-exponent $\nu_2(i)$ of a positive integer i is defined by $i = 2^{\nu_2(i)} j$, where j is an odd integer.

Theorem 1. *Write* $\mathcal{SB}_n = \left[\frac{a_{-1}}{b_{-1}}, \frac{a_0}{b_0}, \frac{a_1}{b_1}, \ldots, \frac{a_{N-1}}{b_{N-1}}\right]$ *where* $N = 2^n$. *Then*

$$a_{-1} = 0,\ a_0 = 1, \qquad a_i = k_i a_{i-1} - a_{i-2} \qquad \text{for } 1 \le i < N, \tag{1a}$$

$$b_{-1} = 1,\ b_0 = n, \qquad b_i = k_i b_{i-1} - b_{i-2} \qquad \text{for } 1 \le i < N, \tag{1b}$$

where $k_i = 2\nu_2(i) + 1$.

Theorem 2. *Write*

$$\mathcal{CW}_\infty = \left[\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{i-1}}{a_i}, \ldots\right]$$

and

$$\mathcal{CW}_n = \left[\frac{b_{-1}}{b_0}, \frac{b_0}{b_1}, \frac{b_1}{b_2}, \ldots, \frac{b_{N-2}}{b_{N-1}}\right]$$

where $N = 2^{n-1}$. *Then the* a_i *and* b_i *can be computed via the recurrence relations*

$$a_{-1} = 0,\ a_0 = 1, \qquad a_i = k_i a_{i-1} - a_{i-2} \qquad \text{for } 1 \le i < \infty, \tag{2a}$$

$$b_{-1} = 1,\ b_0 = n, \qquad b_i = k_i b_{i-1} - b_{i-2} \qquad \text{for } 1 \le i < N, \tag{2b}$$

where $k_i = 2\nu_2(i) + 1$.

Theorem 3. *Write the Farey sequence* $\mathcal{F}_n$ *of order* n *as* $\mathcal{F}_n = \left[\frac{A_{-1}}{B_{-1}}, \frac{A_0}{B_0}, \frac{A_1}{B_1}, \ldots, \frac{A_{N-1}}{B_{N-1}}\right]$. *Then the numerators* A_i *and the denominators* B_i *can be computed via the recurrence relations*

$$A_{-1} = 0,\ A_0 = 1, \qquad A_i = K_i A_{i-1} - A_{i-2} \qquad \text{for } 1 \le i < N, \tag{3a}$$

$$B_{-1} = 1,\ B_0 = n, \qquad B_i = K_i B_{i-1} - B_{i-2} \qquad \text{for } 1 \le i < N, \tag{3b}$$

where $K_i = \left\lfloor \frac{B_{i-2}+n}{B_{i-1}} \right\rfloor$, *and* $N = \sum_{j=1}^n \varphi(j)$.

To illustrate Theorem 1, $\mathcal{SB}_4$ can be computed from left to right using Table 1. Note that the numbers k_i are the same as the numbers k_i' generated by the recurrence $k_1' = 1, k_{2j+1}' = 1, k_{2j}' = k_j' + 2$ for $j \ge 0$. (Proof by induction: $k_1 = k_1'$ and $k_{2j+1} = 1$, $k_{2j} = k_j + 2$ hold for $j \ge 1$ as $\nu_2(2j+1) = 0$ and $\nu_2(2j) = \nu_2(j) + 1$. Thus $k_i = k_i'$ for all $i \ge 1$.)

Table 1. Computing $\mathcal{SB}_4$ from left to right.

i	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_i	0	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1
b_i	1	4	3	5	2	5	3	4	1	3	2	3	1	2	1	1	0
k_i			1	3	1	5	1	3	1	7	1	3	1	5	1	3	1

Proof of Theorem 1. Our proof uses induction on n. It suffices to prove (1a) as the proof of (1b) is similar (just change the as to bs). Clearly (1a) is true for $n = 0$ as $\mathcal{SB}_0 = \left[\frac{0}{1}, \frac{1}{0}\right]$. Assume $n > 0$ and (1a) is true for $\mathcal{SB}_{n-1}$. Let $\frac{a'_{-1}}{b'_{-1}}, \frac{a'_0}{b'_0}, \dots, \frac{a'_{N/2-1}}{b'_{N/2-1}}$ be the fractions in $\mathcal{SB}_{n-1}$. The way mediants are inserted to create $\mathcal{SB}_n$ is shown in Figure 2. Dotted lines denote the repetition of a fraction, and dashed lines denote the formation of a mediant. The repetition of fractions means

$$a_{2j-1} = a'_{j-1} \quad \text{and} \quad b_{2j-1} = b'_{j-1} \quad \text{for } 0 \le j < N/2, \tag{4}$$

and the formation of mediants means

$$a_{2j} = a'_{j-1} + a'_j \quad \text{and} \quad b_{2j} = b'_{j-1} + b'_j \quad \text{for } 0 \le j < N/2. \tag{5}$$

Certainly the formulas for a_{-1} and a_0 in (1a) are correct, as $\mathcal{SB}_n$ starts with $\frac{0}{1}, \frac{1}{n}$. Suppose now that $1 \le i < N$, and consider the cases when i is even and odd.

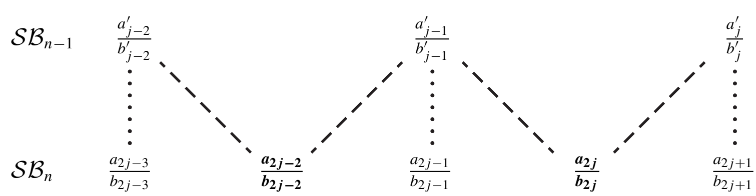

Figure 2. Constructing $\mathcal{SB}_n$ from $\mathcal{SB}_{n-1}$ by inserting mediants.

Case 1. $i = 2j$ is even and $j \ge 1$. The following shows that (1a) holds for even i:

$$\begin{aligned}
k_{2j}a_{2j-1} - a_{2j-2} &= (k_j + 2)a_{2j-1} - a_{2j-2} && \text{as } k_{2j} = k_j + 2, \\
&= (k_j + 2)a'_{j-1} - (a'_{j-2} + a'_{j-1}) && \text{by (4) and (5),} \\
&= k_j a'_{j-1} - a'_{j-2} + a'_{j-1} && \text{canceling } a'_{j-1}, \\
&= a'_j + a'_{j-1} && \text{as } a'_j = k_j a'_{j-1} - a'_{j-2} \text{ by induction,} \\
&= a_{2j} && \text{by (5).}
\end{aligned}$$

Case 2. $i = 2j + 1$ is odd and $j \ge 1$. It follows from $k_{2j+1} = 1$ and (4) and (5) that

$$k_{2j+1}a_{2j} - a_{2j-1} = a_{2j} - a_{2j-1} = (a'_{j-1} + a'_j) - a'_{j-1} = a'_j = a_{2j+1},$$

as desired. This completes the induction on n. ■

A different (and very interesting) method for computing terms of $\mathcal{SB}_n$ is given in [**1**]. It uses continued fraction expansions and "normalized additive factorizations." As the recurrence (1a) is independent of n, the numerators for $\mathcal{SB}_{n-1}$ reappear as the first $2^{n-1} + 1$ numerators for $\mathcal{SB}_n$. We now show that (half of) the denominators b_i in $\mathcal{SB}_n$ reappear (remarkably!) in $\mathcal{CW}_n$, and the numerators a_i also reappear in $\mathcal{CW}_\infty$. Accordingly, we have used the *same notation* a_i, b_i in Theorem 2 as in Theorem 1.

Proof of Theorem 2. Figure 3 shows that the numbers a_i must satisfy the recurrence relation:

$$a_0 = 1, \quad a_{2j-1} \overset{(6.1)}{=} a_{j-1} \quad \text{and} \quad a_{2j} \overset{(6.2)}{=} a_{j-1} + a_j \quad \text{for } j > 0. \tag{6}$$

Define $a_{-1} = 0$. With this definition (6.1) and (6.2) also hold when $j = 0$. We shall now prove, by induction on i, that the formulas for a_i in (2a) are correct. This is clear for $i = -1, 0$. Now suppose $i > 0$, and assume the formulas in (2a) hold for all $i' < i$.

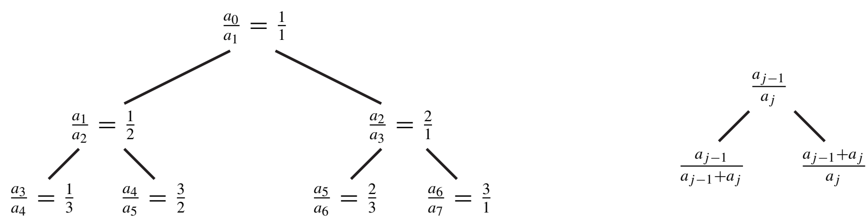

Figure 3. C-W tree (left), and C-W rules (right).

CASE 1. $i = 2j$ where $j \geq 1$. Then

$$\begin{aligned} k_{2j}a_{2j-1} - a_{2j-2} &= (k_j + 2)a_{2j-1} - a_{2j-2} && \text{as } k_{2j} = k_j + 2, \\ &= (k_j + 2)a_{j-1} - (a_{j-2} + a_{j-1}) && \text{by (6.1) and (6.2)}, \\ &= k_j a_{j-1} - a_{j-2} + a_{j-1} && \text{canceling } a_{j-1}, \\ &= a_j + a_{j-1} && \text{by induction on } i, \\ &= a_{2j} && \text{by (6.1)}. \end{aligned}$$

CASE 2. $i = 2j - 1$ where $j \geq 1$. Then $k_{2j+1} = 1$, so (6) implies

$$k_{2j+1}a_{2j} - a_{2j-1} = a_{2j} - a_{2j-1} = (a_{j-1} + a_j) - a_{j-1} = a_j = a_{2j+1}.$$

This completes the inductive proof of (2a).

The proof of (2b) is now straightforward. The first term $\frac{b_{-1}}{b_0} = \frac{1}{n}$ of $\mathcal{CW}_n$ equals $\frac{a_{N-1}}{a_N}$ where $N = 2^{n-1}$. As $\mathcal{CW}_n$ is a consecutive subsequence of $\mathcal{CW}_\infty$, it follows that $a_{N+i} = b_i$ for $-1 \leq i < N$. Now $a_{N-1} = 1, a_N = n$, and (2a) implies that $a_{N+i} = k_{N+i}a_{N+i-1} - a_{N+i-2}$ for $1 \leq i < N$. However, $\nu_2(N + i) = \nu_2(i)$ and $k_{N+i} = k_i$ for $1 \leq i < N$. Thus (2b) follows by replacing a_{N+i} with b_i, and k_{N+i} with k_i, for $1 \leq i < N$. ■

Theorem 3 is previously known (see Exercise 4.61 in [**3**, p. 150]). We include Theorem 3 and its proof both for comparison with Theorems 1 and 2, and for the reader's convenience.

Proof of Theorem 3. As the first two fractions of $\mathcal{F}_n$ are $\frac{0}{1}$ and $\frac{1}{n}$, the recurrences (3a, b) are correct for $i = -1, 0$. Now suppose $1 \leq i < N$. By definition, $\frac{A_{i-2}}{B_{i-2}}$, $\frac{A_{i-1}}{B_{i-1}}$, and $\frac{A_i}{B_i}$ are consecutive terms in $\mathcal{F}_n$. We shall prove that $A_i = a$ and $B_i = b$, where

$a = K_i A_{i-1} - A_{i-2}$ and $b = K_i B_{i-1} - B_{i-2}$. We know that $A_{i-1}B_{i-2} - B_{i-1}A_{i-2} = 1$ since consecutive Farey fractions are adjacent. Hence

$$\begin{aligned} aB_{i-1} - bA_{i-1} &= (K_i A_{i-1} - A_{i-2})B_{i-1} - (K_i B_{i-1} - B_{i-2})A_{i-1} \\ &= A_{i-1}B_{i-2} - B_{i-1}A_{i-2} = 1. \end{aligned} \tag{7}$$

Consider the inequalities $\frac{B_{i-2}+n}{B_{i-1}} - 1 < K_i \leq \frac{B_{i-2}+n}{B_{i-1}}$. Multiplying by B_{i-1} and subtracting B_{i-2} gives $n - B_{i-1} < b \leq n$. It follows from $B_{i-1} \leq n$ that $0 < b \leq n$, and it follows from (7) that $\frac{A_{i-2}}{B_{i-2}} < \frac{A_{i-1}}{B_{i-1}} < \frac{a}{b}$. Since $\frac{A_i}{B_i}$ is, by definition, the next fraction in $\mathcal{F}_n$ after $\frac{A_{i-1}}{B_{i-1}}$, we conclude that $\frac{A_{i-1}}{B_{i-1}} < \frac{A_i}{B_i} \leq \frac{a}{b}$.

We must show that $\frac{A_i}{B_i} = \frac{a}{b}$. Suppose to the contrary that $\frac{A_i}{B_i} < \frac{a}{b}$. Then

$$1 \overset{(8.1)}{\leq} aB_i - bA_i \qquad \text{and} \qquad 1 \overset{(8.2)}{=} A_i B_{i-1} - B_i A_{i-1}. \tag{8}$$

Multiplying (8.1) by B_{i-1}, and (8.2) by b, and then adding gives

$$n < B_{i-1} + b \leq (aB_i - bA_i)B_{i-1} + b(A_i B_{i-1} - B_i A_{i-1}) = (aB_{i-1} - bA_{i-1})B_i \overset{(7)}{=} B_i.$$

This is a contradiction since $\frac{A_i}{B_i} \in \mathcal{F}_n$ has $B_i \leq n$. Hence $\frac{A_i}{B_i} = \frac{a}{b}$. As both fractions are reduced (and $A_i, B_i, b > 0$), we conclude that $A_i = a$ and $B_i = b$, as desired.

3. CONCLUDING REMARKS. The On-Line Encyclopedia of Integer Sequences [**6**] has a wealth of useful information about the sequences $a_0, a_1, a_2, \ldots$ (A002487) and $k_1, k_2, k_3, \ldots$ (A037227). However, the connection in Theorem 2 between these sequences is new. Note that a_n counts the number of ways that n can be written as a sum of powers of 2, each power being used at most twice. For example, $a_4 = 3$ as $2^2 = 2 + 2 = 2 + 1 + 1$.

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