Math 1B Project 3
Continued Fractions.
Many of the special functions that occur in the applications of mathematics are defined by infinite processes, such as series, integrals, and iterations. The continued fraction is one of these infinite processes. An example of a continued fraction is this one, due to Lambert in 1770:

$$\arctan x = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{4x^2}{5+\cfrac{9x^2}{7+\cfrac{16x^2}{9+\cdots}}}}} \quad (|x|<1)$$

This can also be written as

$$\arctan x = \frac{x}{1+}\frac{x^2}{3+}\frac{4x^2}{5+}\frac{9x^2}{7+}\frac{16x^2}{9+}\cdots \quad (|x|<1) \tag{1}$$

The right side of this equation represents a limit in the following way: The expression in Equation (2) is terminated after n terms to give a will-defined function f_n:

$$f_n(x) = \frac{x}{1+}\frac{x^2}{3+}\frac{4x^2}{5+}\frac{9x^2}{7+}\frac{16x^2}{9+}\cdots\frac{(n-1)^2x^2}{2n-1} \quad (n\geq 2) \tag{2}$$

This is called the nth *convergent* of the continued fraction. Equation (2) is defined to mean

$$\arctan x = \lim_{x\to\infty} f_n(x) \quad (|x|<1)$$

We haven't proved this, but it can be proved (trust...but verify) and so we have an alternative method of computing the values of the arctan() function. Whether the method is practical depends on the *rapidity* of convergence in Equation (4). To judge this numerically, let us compute $\arctan(1/\sqrt{3}) = \pi/6 \approx 0.5235987756$ by means of the sequence $f_n(1/\sqrt{3})$ for $n \geq 2$. Here are the results:

n	$f_n(1/\sqrt{3})$
2	0.519615
3	0.523892
4	0.523577
5	0.523600
6	0.523599
7	0.523599

This list shows that six decimal places of precision have been obtained at the sixth convergent.

1 Recursive Formulas

The task of evaluated a continued fraction is not as easy as evaluating a series. In the case of an infinite series, say $\sum\_k = 1^\infty a_k$, we compute the partial sums $S_n \sum_{k=1}^{n} a_k$ by means of the formula $S_{n+1} = S_n + a_{n+1}$. Now consider the analogous problem for a continued fraction:

$$C = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots \tag{3}$$

We want to discover a recursive algorithm for computing the successive convergents:

$$C_n = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_{n-1}}{b_{n-1}+}\frac{a_n}{b_n} \tag{4}$$

Consider the function,

$$f_b(x) = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_{n-1}}{b_{n-1}+}\frac{a_n}{b_n+x} \tag{5}$$

Obviously then

$$C_n = f_n(0) \tag{6}$$

To get iterates $f_n(x)$ from $f_{n-1}(x)$, observe that

$$f_n(x) = f_{n-1}\left(\frac{a_n}{b_n + x}\right) \tag{7}$$

By simplifying the subsequent complex fractions to simple fractions, we have

$$f_1(x) = \frac{a_1}{b_1 + x} \tag{8}$$

$$f_2(x) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + x}} = \frac{a_1 b_2 + a_1 x}{b_1 b_2 + a_2 + b_1 x} \tag{9}$$

$$f_3(x) = \frac{a_1 b_2 + a_1 x}{b_1 b_2 + a_2 + b_1 \frac{a_3}{b_3 + x}} = \frac{a_1 b_2 b_3 a_1 a_3 + (a_1 b_2)x}{b_1 b_2 b_3 + a_2 b_3 + b_1 a_3 + (b_1 b_2 + a_2)x} \tag{10}$$

This suggests the pattern

$$f_n(x) = \frac{A_n + A_{n-1}x}{B_n + B_{n-1}x} \quad (n \geq 1) \tag{11}$$

where

$$\begin{cases} A_0 = 0, A_1 = a_1 \\ A_n = b_n A_{n-1} + a_n A_{n-2} \quad (n \geq 2) \end{cases} \tag{12}$$

and

$$\begin{cases} B_0 = 1, B_1 = b_1 \\ B_n = b_n B_{n-1} + a_n B_{n-2} \quad (n \geq 2) \end{cases} \tag{13}$$

THEOREM 1 Theorem on Continued Fractions *If the sequences* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are given, and if the sequences* $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ are defined by the Formulas (12) and (13), *then*

$$C_n = \frac{A_n}{B_n} \quad (\geq 1)$$

Proof The validity of Equation (11) for indices 1, 2, and 3 has been verified. Now make the inductive hypothesis: that Equation (11) is true for indices $1, 2, \ldots, n-1$. Then

$$f_n(x) = f_{n-1}\left(\frac{a_n}{b_n + x}\right) \tag{14}$$

$$= \frac{A_{n-1} + A_{n-2}a_n/(b_n + x)}{B_{n-1} + B_{n-2}a_n/(b_n + x)} \tag{15}$$

$$= \frac{A_{n-1}(b_n + x) + A_{n-2}a_n}{B_{n-1}(b_n + x) + B_{n-2}a_n} \tag{16}$$

$$= \frac{A_{n-1}b_n + A_{n-2}a_n + A_{n-1}x}{B_{n-1}b_n + B_{n-2}a_n + B_{n-1}x} \tag{17}$$

$$= \frac{A_n + A_{n-1}x}{B_n + B_{n-1}x} \tag{18}$$

This establishes Equation (11) by mathematical induction. Then it follows

$$C_n = f_n(0) = \frac{A_n}{B_n}$$

The recursive formulas developed here form the basis for an efficient algorithm. For example, the numerical results for $\arctan(1/\sqrt{3})$, at the beginning of this paper, were computed using this recursive relation with $a_n = (n-1)^2 x^2$ and $b_n = 2n - 1$.

2 Conversion of Series to Continued Fractions

Many important special functions that arise in applied mathematics have expansions in continued fractions. Sources of information are Abramowitz and Stegun [1964] and the textbooks by Khovanskii [1963], Perron [1929], and Wall [1948]. We shall indicate here one procedure by which a continued fraction can be obtained from a series.

THEOREM 2 Theorem on Series to Continued Fractions

$$\sum_{k=1}^{\infty} \frac{1}{x_k} = \frac{1}{x_1-} \frac{x_1^2}{x_1+x_2-} \frac{x_2^2}{x_2+x_3-} \cdots \frac{x_{n-1}^2}{x_{n-1}+x_n-} \cdots \tag{19}$$

Proof This can be proved by induction and is left as an exercise. To illustrate how this theorem can be used, we construct a continued fraction from the Maclaurin series for $\arctan x$:

$$\begin{aligned} \arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \\ &= \frac{1}{x^{-1}} + \frac{1}{-3x^{-3}} + \frac{1}{5x^{-5}} + \frac{1}{-7x^{-7}} + \cdots \\ &= \frac{1}{x^{-1}-} \frac{(x^{-1})^2}{x^{-1} + (-3x^{-3})} - \frac{(-3x^{-3})^2}{(-3x^{-3}) + (5x^{-5})-} \frac{(5x^{-5})^2}{(5x^{-5}) + (-7x^{-7})-} \cdots \\ &= \frac{x}{1+} \frac{-x^2}{x^2 - 3+} \frac{-9x^2}{-3x^2 + 5+} \frac{-25x^2}{5x^2 - 7-} \cdots \end{aligned}$$

Notice that if both the numerator and denominator of one of the component fractions in a continued fraction are multiplied by the same quantity, then the numerator of the next component fraction is affected as well. (Why?) Equations (12), (13), and (14) can be used to approximate $\arctan x$ for a given value of x when

$$a_n = -(2n-3)^2 x^2 \quad \texttt{and} \quad b_n = (-1)^n[(2n-3)x^2 - (2n-1)]$$

for $n \geq 2$. However, the algorithm for computing $\arctan x$ via this continued fraction is far more complicated than one based on computing the partial sums in the Maclaurin series, and it produces the same sequence of numbers. Observe that the Lambert continued fraction for $\arctan x$, in Equation(1), is not the same as the one derived from the Maclaurin series. To investigate this further, we compute $arctan(1/\sqrt{3})$ using the partial sums in the series:

n	$f_n(1/\sqrt{3})$
1	0.577350
2	0.513200
3	0.526030
4	0.522976
5	0.523767
6	0.523551
7	0.523612
8	0.523595
9	0.523600
10	0.523598
11	0.523599
12	0.523599

A comparison of this table with the one produced from the continued fraction(1) shows that the continued fraction converges faster than the series, for this example.

1. Show that

$$\frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \cdots = \frac{1}{2}(\sqrt{5} - 1)$$

Hint: Set x equal to the continued fraction and look at $1/x$.

2. (Continuation) for the continued fraction in the preceding problem, show that

$$C_n = \frac{A_n}{A_{n+1}}$$

3. Show that

$$\sqrt{x} = 1 + 2\left(\frac{v}{1+}\frac{v}{1+}\frac{v}{1+}\cdots\right)$$

where $v = \frac{1}{4}(x-1)$.

4. (Continuation) For the continued fraction in the preceding problem, show that

$$C_n = \frac{vA_n}{A_{n+1}}$$

5. Show that

$$\sqrt{b^2+a} = b + \frac{a}{2b+}\frac{a}{2b+}\frac{a}{2b+}\cdots$$

6. Compare two methods for computing values of the function

$$f(x) = \int_0^x e^{-t^2}\,dt$$

namely, the Taylor series expansion and the continued fraction.

$$f(x) = \frac{\sqrt{\pi}}{2} - \frac{1}{2}e^{-x^2}\left(\frac{1}{x+}\frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\frac{4}{x+}\cdots\right)$$

7. Show that every real number in the interval $0 < x < 1$ can be expressed as a continued fraction (possibly terminating) in the form

$$x = \frac{1}{b_1+}\frac{1}{b_2+}\frac{1}{b_3+}\cdots$$

where each b_i is a positive integer

8. Prove that if

$$x = \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots$$

with the a_i and b_i all positive, then the convergents of the fraction are alternately greater than and less than x.

9. Show that f_n is increasing if and only if $A_{n-1}B_n > A_nB_{n-1}$, using the notation of Equation (11).

10. Count the number of long operations involved in computing C_n from A_n/B_n by the use of Formulas (12) and (13).

11. Show that any two successive convergents in the continued fraction of Equation (14) obey this equation:

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = (-1)^{n-1}\frac{a_1a_2\cdots a_n}{B_nB_{n-1}}$$

12. (Continuation) Use the result of the preceding problem to prove Problem (8) above.

13. Prove that if the numbers b_i are all positive and $a_i = 1$, then $B_n > \min(1, b_1)$ for all n.

14. Prove that if $b_i > 0$ and $a_i = 1$, then B_nB_{n-1} is increasing as a function of n.

15. Prove that if $b_i > 1 = a_i$, then the continued fraction in Equation (14) converges.

16. Find a recrusive formula for B_n if

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots = \frac{1}{1+}\frac{1}{B_2+}\cdots$$

17. Prove Theorem 2 in this section. *Hint:* Use induction.

18. Show that

$$e^x = \frac{1}{1-}\frac{x}{x+1-}\frac{x}{x+2-}\frac{2x}{x+3-}\frac{3x}{x+4-}\cdots$$

19. (Continuation) Show that

$$\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\cdots = \frac{1}{e-1}$$

Hint Use the preceding problem.

20. Show that the functions

$$g_n(x) = \frac{1}{x+}\frac{2}{x+}\frac{3}{x+}\cdots\frac{n}{x}$$

cna be generated by this algorithm:

$$g_n(x) = \frac{p_n(x)}{q_n(x)}$$

where

$$\begin{cases} p_0(x) = 0, p_1(x) = 1 \\ p_{n+1}(x) = xp_n(x) + (n+1)p_{n-1}(x) \end{cases}$$

and

$$\begin{cases} q_0(x) = 1, q_1(x) = x \\ q_{n+1}(x) = xq_n(x) + (n+1)q_n(x) \end{cases}$$

21. Assuming that the following continued fraction converges, find its value:

$$\frac{1}{6+}\frac{1}{6+}\frac{1}{6+}\frac{1}{6+}\cdots$$

22. Find the value of x:

$$x = 1 + \frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}$$

23. Find the value of x:

$$x = 2 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+}\frac{1}{4+}$$

24. If

$$f_n(x) = \frac{2}{1+}\frac{4}{2+}\frac{6}{3+}\frac{2n}{n+x}$$

how would you obtain f_{n+1} from $f_n(x)$?

25. What is the value of $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$?

26. Assume that the continued fraction

$$\frac{1}{2x+}\frac{1}{2x+}\frac{1}{2x+}\cdots \quad (x > 0)$$

converges. Determine a closed-form expression for it in terms of x.

Computer Problems.

27. Write a program for computing $\sqrt{x}$ by means of the equation in problem 3. Compute $\sqrt{10}$,$\sqrt{100}$,$\sqrt{1000}$, and $\sqrt{10000}$ by printing a table of the first 50 convergent values for each.

28. Write a prgram for computing $arctan(1/\sqrt{3})$ without using subscripted variables, and compare to the rsults given in the text.

29. Write a computer program to evaluate the continued fraction for *arctanx*, as given in Equation (1). Use the following elementary approach: Given n, compute $f_n(x)$ in Equation (3) by starting at the right side and forming the appropriate fractions. Test your program by computing $\pi^{-1}\arctan(\sqrt{3})$ with $n = 5, 10, 15$, and 20.